Introduction to rotordynamics

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Outline

1 Introduction
   Structures of interest
   Mechanical components
   Selected topics
   History and scientists

2 Equations of motion
   Inertial and moving frames
   Displacements and velocities
   Strains and stresses
   Energies and virtual works
   Displacement discretization

3 Structural analysis
   Static equilibrium
   Modal analysis

4 Case studies
   Case 1: flexible shaft bearing system
   Case 2: bladed disks
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Structures of interest

Jet engines

Machine tools

Electricity power plants

Gyroscopes
Mechanical components

Terminology

- Supporting components
- Driving components
- Operating components

Definitions

- Supporting components: journal bearings, seals, magnetic bearings
- Driving components: shaft
- Operating components: bladed disk, machine tools, gears
1 Driving components
   - linear vs. nonlinear formulations
   - constant vs. non-constant rotational velocities $\Omega$
   - $\Omega$-dependent eigenfrequencies
   - isotropic vs. anisotropic cross-sections
   - rotational vs. reference frames
   - stationary and rotating dampings
   - gyroscopic terms
   - external forcings and imbalances
   - critical rotational velocities, Campbell diagrams, stability
   - reduced-order models and modal techniques
   - fluid-structure coupling
   - bifurcation analyses, chaos

2 Supporting components
   - linear vs. nonlinear formulations
   - constant vs. non-constant rotational velocities
   - gyroscopic terms
   - fluid-structure coupling

3 Operating components
   - linear vs. nonlinear formulations
   - concept of axi-symmetry
   - reduced-order models
   - fluid-structure coupling
   - critical rotational velocities, Campbell diagrams, stability
   - fixed vs. moving axis of rotation
   - centrifugal stiffening
History and scientists

- 1869 – Rankine – *On the centrifugal force on rotating shafts*
  - steam turbines
  - notion of critical speed

- 1895 – Föppl, 1905 – Belluzo, Stodola
  - notion of supercritical speed

- 1919 – Jeffcott – *The lateral vibration of loaded shafts in the neighborhood of a whirling speed*

- Before WWII: Myklestadt-Prohl method
  - lumped systems
  - cantilever aircraft wings
  - precise computation of critical speeds

- Contemporary tools
  - finite element method
  - nonlinear analysis
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Inertial and moving frames

Full 3D rotating motion of the dynamic frame
- Fixed (galilean, inertial) reference frame $R_g$
- Moving (non-inertial) reference frame $R_m$
- Rigid body large displacements $R_m/R_g$
- Small displacements and strains in $R_m$

Coordinates
- $x$: position vector of $M$ in $R_m$
- $u$: displacement vector of $M/R_m$ in $R_m$
- $s$: displacement vector of $O'/R_g$ in $R_g$
- $y$: displacement vector of $M/R_g$ in $R_g$
Displacements and velocities

Rotation matrices

\[
R_1 = \begin{bmatrix}
\cos \theta_z & -\sin \theta_z & 0 \\
\sin \theta_z & \cos \theta_z & 0 \\
0 & 0 & 1 \\
\end{bmatrix};
\quad R_2 = \begin{bmatrix}
\cos \theta_Y & 0 & -\sin \theta_Y \\
0 & 1 & 0 \\
\sin \theta_Y & 0 & \cos \theta_Y \\
\end{bmatrix};
\quad R_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_1 & -\sin \theta_1 \\
0 & \sin \theta_1 & \cos \theta_1 \\
\end{bmatrix}
\]

\[
R = R_3 R_2 R_1
\]

Coordinates transformation

- absolute displacement \( y = s + R(x + u) \)
- angular velocity matrix \( \dot{\Omega} | \dot{R} = R \Omega \)

Velocities

- Time derivatives of positions in \( R_g \)
- \( \dot{y} = \dot{s} + \dot{R}u + \dot{R}(x + u) = \dot{s} + \dot{R}u + R \Omega (x + u) \)

\[
\Omega = \begin{bmatrix}
0 & -\Omega_3(t) & \Omega_2(t) \\
\Omega_3(t) & 0 & -\Omega_1(t) \\
-\Omega_2(t) & \Omega_1(t) & 0 \\
\end{bmatrix}
\]
Quantities of interest

- Stresses $\sigma$
- Strains $\varepsilon$: Green measure

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right) = \varepsilon^{\text{ln}}_{ij} + \varepsilon^{\text{nl}}_{ij}$$

- Pre-stressed configuration and centrifugal stiffening

Constitutive law

- Generalized Hooke law $\sigma = C \varepsilon$
- $C \rightarrow$ linear elasticity tensor
Energies and virtual works

Kinetic energy

\[
E_K = \frac{1}{2} \int_V \rho \dot{y}^T \dot{y} \, dV = \frac{1}{2} \int_V \rho \dot{u}^T \dot{u} \, dV + \int_V \rho \dot{u}^T \Omega u \, dV + \frac{1}{2} \int_V \rho u^T \Omega^2 u \, dV \\
- \int_V \rho (u^T \Omega - \dot{u}^T)(R^T \dot{s} + \Omega x) \, dV + \frac{1}{2} \int_V \rho (\dot{s}^T \dot{s} + 2\dot{s}^T R \Omega x - x \Omega^2 x) \, dV
\]

Strain energy

\[
U = \frac{1}{2} \int_V \varepsilon^T C \varepsilon \, dV
\]

External virtual works

\[
W_{\text{ext}} = \int_V u^T Rf \, dV + \int_S u^T R \mathbf{t} \, dS \quad \leftarrow (f, \mathbf{t}) \text{ expressed in } Rm
\]
Displacement discretization

Discretized displacement field

- Rayleigh-Ritz formulation
- Finite Element formulation

\[ u(x, t) = \sum_{i=1}^{n} \Phi_i(x) u_i(t) \]
Displacement discretization

Generic equations of motion

\[ M \ddot{u} + (D + G(\Omega)) \dot{u} + \left( K(u) + P(\dot{\Omega}) + N(\Omega) \right) u = R(\Omega) + F \]

Structural matrices

- mass matrix \( M = \int_V \rho \Phi^T \Phi \, dV \)
- gyroscopic matrix \( G = \int_V 2\rho \Phi^T \Omega \Phi \, dV \)
- centrifugal stiffening matrix \( N = \int_V \rho \Phi^T \Omega^2 \Phi \, dV \)
- angular acceleration stiffening matrix \( P = \int_V \rho \Phi^T \dot{\Omega} \Phi \, dV \)
- stiffness matrix \( K = \int_V \sigma^T \varepsilon \, dV = \int_V \varepsilon^T (u) C \varepsilon (u) \, dV \) ← nonlinear in \( u \)
- damping matrix \( D \) (several different definitions)

External force vectors

- inertial forces \( R = -\int_V \rho \Phi^T (R^T s + \dot{\Omega} x + \Omega^2 x) \, dV \)
- external forces \( F = \int_V \Phi^T f \, dV + \int_{S_F} \Phi^T t \, dS \)
Displacement discretization

Generic equations of motion

\[ M\ddot{u} + (D + G(\Omega))\dot{u} + \left( K(u) + P(\dot{\Omega}) + N(\Omega) \right) u = R(\Omega) + F \]

Choice space functions \( \Phi \) and time contributions \( u \)

- order-reduced models or modal reductions
- 1D, 2D or 3D elasticity
- beam and shell theories (Euler-Bernoulli, Timoshenko, Reissner-Mindlin…)
- complex geometries: bladed-disks

Other simplifying assumptions

- constant velocity along a single axis of rotation
- no gyroscopic terms
- no centrifugal stiffening
- localized vs. distributed imbalance
- isotropy or anisotropy of the shaft cross-section
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Static equilibrium

Solution and pre-stressed configuration

Find displacement $u^0$, solution of:

$$
(K(u^0) + N(\Omega)) u^0 = R(\Omega) + F
$$

Applications

- slender structures
- plates, shells
- shaft with axial loading
- radial traction and blade untwist
Modal analysis

Eigensolutions

- Defined with respect to the pre-stressed configuration $V^*$
- State-space formulation

\[
\begin{bmatrix}
M & D + G(\Omega) \\
0 & -M
\end{bmatrix}
\begin{bmatrix}
\ddot{u} \\
\dot{u}
\end{bmatrix}
+ \begin{bmatrix}
0 & K(u^0) + N(\Omega) \\
M & 0
\end{bmatrix}
\begin{bmatrix}
\dot{u} \\
u
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  

(1)

- Consider:

\[y = \begin{bmatrix}
\dot{u} \\
u
\end{bmatrix} = Ae^{\lambda t} \text{ and find } (y, \lambda) \text{ solution of (1)}\]

Comments

- $\Omega$-dependent complex (conjugate) eigenmodes and eigenvalues
- Right and left eigenmodes
- Whirl motions
- **Campbell diagrams and stability conditions**
Modal analysis

Campbell diagrams

- Coincidence of vibration sources and $\omega$-dependent natural resonances
- Quick visualisation of vibratory and design problems

Stability

$$\exists \lambda_i \mid \Re(\lambda_i) > 0 \Rightarrow \text{unstable}$$
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Case 1: flexible shaft bearing system

Rotating shaft bearing system
- Constant velocity
- Small deflections
- Elementary slice $\Leftrightarrow$ rigid disk

Quantities of interest
- Kinetic Energy
  $$E_K = \frac{\rho S}{2} \int_0^L (\dot{u}^2 + \dot{w}^2) dy + \frac{\rho I}{2} \int_0^L \left[ \left( \frac{\partial \dot{u}}{\partial y} \right)^2 + \left( \frac{\partial \dot{w}}{\partial y} \right)^2 \right] dy + \rho I L \Omega^2 - 2 \rho I \Omega \int_0^L \frac{\partial \dot{u}}{\partial y} \frac{\partial \dot{w}}{\partial y} dy$$
- Strain Energy
  $$U = \frac{EI}{2} \int_0^L \left[ \left( \frac{\partial^2 u}{\partial y^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 \right] dy$$
- Virtual Work of external forces
  $$\delta W = F \delta u + G \delta w$$
Case 1: flexible shaft bearing system

Reynolds’ equation
- $h$ journal/bearing clearance
- $\mu$ fluid viscosity
- $\Omega$ angular velocity of the journal
- $R$ outer radius of the beam
- $p$ film pressure

\[
\frac{\partial}{\partial y} \left( \frac{h^3}{6\mu} \frac{\partial p}{\partial y} \right) + \frac{1}{R^2} \frac{\partial}{\partial \theta} \left( \frac{h^3}{6\mu} \frac{\partial p}{\partial \theta} \right) = \Omega \frac{\partial h}{\partial \theta} + 2 \frac{\partial h}{\partial t}
\]

Bearing forces
- $H_j = 1 - Z_j \cos(\theta + \phi) + X_j \sin(\theta + \phi)$
- $G = Z_j \sin(\theta + \phi) + X_j \cos(\theta + \phi) - 2(\dot{Z}_j \cos(\theta + \phi) - \dot{X}_j \sin(\theta + \phi))$

\[
F_x = -\frac{\mu RL_b^3 \Omega}{2c^2} \int_0^\pi \frac{G \sin(\theta + \phi)}{H_j^3} d\theta; \\
F_z = \frac{\mu RL_b^3 \Omega}{2c^2} \int_0^\pi \frac{G \cos(\theta + \phi)}{H_j^3} d\theta
\]
Case 1: flexible shaft bearing system

Linearization

- Equilibrium position $X_e, Z_e$
- First order Taylor series of the nonlinear forces

$$K_{XX} = \left. \frac{\partial F_x}{\partial X} \right|_e; \quad K_{XZ} = \left. \frac{\partial F_x}{\partial Z} \right|_e; \quad K_{ZX} = \left. \frac{\partial F_z}{\partial X} \right|_e; \quad K_{ZZ} = \left. \frac{\partial F_z}{\partial Z} \right|_e$$

$$C_{XX} = \left. \frac{\partial F_x}{\partial \dot{X}} \right|_e; \quad C_{XZ} = \left. \frac{\partial F_x}{\partial \dot{Z}} \right|_e; \quad C_{ZX} = \left. \frac{\partial F_z}{\partial \dot{X}} \right|_e; \quad C_{ZZ} = \left. \frac{\partial F_z}{\partial \dot{Z}} \right|_e$$

Field of displacement

$$U(x, t) = \sum_{i=1}^{N} \phi_i(x) u_i(t); \quad W(x, t) = \sum_{i=1}^{N} \phi_i(x) w_i(t) \quad \Rightarrow \quad x = (u, w)^T$$

Linear equations of motion

$$M\ddot{x} + (G + C_b)\dot{x} + (K + K_b)x = 0$$
Case 1: flexible shaft bearing system

Linear equations of motion

\[
M\ddot{x} + (\Omega G + C_b(\Omega))\dot{x} + (K + K_b(\Omega))x = F_{NL}
\]

\[
M = \begin{bmatrix} A + B & 0 \\ 0 & A + B \end{bmatrix} ; \quad G = \begin{bmatrix} 0 & -2B \\ 2B & 0 \end{bmatrix} ; \quad K = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}
\]

\[
A = \rho S \int_0^L \Phi^T \Phi \, dy; \quad B = \rho l \int_0^L \Phi_y^T \Phi_y \, dy; \quad C = E l \int_0^L \Phi_{yy}^T \Phi_{yy} \, dy
\]

Comments

- \( K_b \) circulatory matrix \( \rightarrow \) destabilizes the system
- \( C_b \) symmetric damping matrix
- both matrices act on the shaft at the bearing locations (external virtual work)
Case 1: flexible shaft bearing system

Modal analysis

- Equilibrium position for each $\Omega$
- $C_b$, $K_b$ for each $\Omega$
- Modal analysis for each $\Omega$

Stability: linear vs. nonlinear models
Case 1: flexible shaft bearing system

Modal analysis

- Equilibrium position for each $\Omega$
- $C_b$, $K_b$ for each $\Omega$
- Modal analysis for each $\Omega$

Stability: linear vs. nonlinear models
Case 1: flexible shaft bearing system

A few modes

- cylindrical modes

- conical modes
Case 2: bladed disks

Definition

- Closed replication of a sector
- Cyclic-symmetry

Mathematical properties

- Invariant with respect to the axis of rotation
- Block-circulant mass and stiffness matrices
- Pairs of orthogonal modes with identical frequency

Numerical issues

- No damping and gyroscopic matrices
- Determination of $\mathbf{M}$ and $\mathbf{K}$
- Notion of mistuning
- Constant $\Omega$
Case 2: bladed disks

Multi-stage configuration

- Closed replication of a "slice"
- Different number of blades
- No cyclic-symmetry

Numerical issues

- No damping and gyroscopic matrices
- Determination of $M$ and $K$
- Notion of mistuning
- Constant $\Omega$

Next-generation calculations

- Nonlinear terms
- Supporting+transmission+operating components
- Damping, gyroscopic terms, stiffening
- Time-dependent $\Omega$
References

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