# Introduction to rotordynamics

Mathias Legrand

McGill University

Structural Dynamics and Vibration Laboratory

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# Outline

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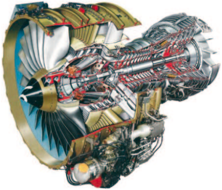
# Structures of interest

## Jet engines



## Machine tools

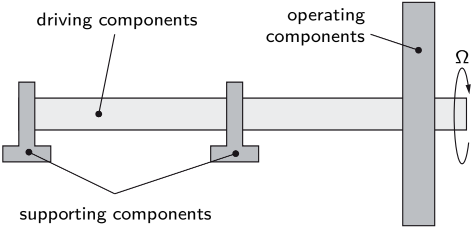
## Electricity power plants

## Gyroscopes

# Mechanical components

## Terminology

- Supporting components
- Driving components
- Operating components




## Definitions

- Supporting components: journal bearings, seals, magnetic bearings
- Driving components: shaft
- Operating components: bladed disk, machine tools, gears

# Selected topics

1. **Driving components**
   - linear vs. nonlinear formulations
   - constant vs. non-constant rotational velocities $\Omega$
   - $\Omega$-dependent eigenfrequencies
   - isotropic vs. anisotropic cross-sections
   - rotational vs. reference frames
   - stationary and rotating dampings
   - gyroscopic terms
   - external forcings and imbalances
   - critical rotational velocities, CAMPBELL diagrams, stability
   - reduced-order models and modal techniques
   - fluid-structure coupling
   - bifurcation analyses,chaos
2. **Supporting components**
   - linear vs. nonlinear formulations
   - constant vs. non-constant rotational velocities
   - gyroscopic terms
   - fluid-structure coupling
3. **Operating components**
   - linear vs. nonlinear formulations
   - concept of axi-symmetry
   - reduced-order models
   - fluid-structure coupling
   - critical rotational velocities, CAMPBELL diagrams, stability
   - fixed vs. moving axis of rotation
   - centrifugal stiffening

# History and scientists

- 1869 – RANKINE – *On the centrifugal force on rotating shafts*
  - steam turbines
  - notion of critical speed
- 1895 – FÖPPL, 1905 – BELLUZO, STODOLA
  - notion of supercritical speed
- 1919 – JEFFCOTT – *The lateral vibration of loaded shafts in the neighborhod of a whirling speed*
- Before WWII: MYKLESTADT-PROHL method
  - lumped systems
  - cantilever aircraft wings
  - precise computation of critical speeds
- Contemporary tools
  - finite element method
  - nonlinear analysis

# Outline

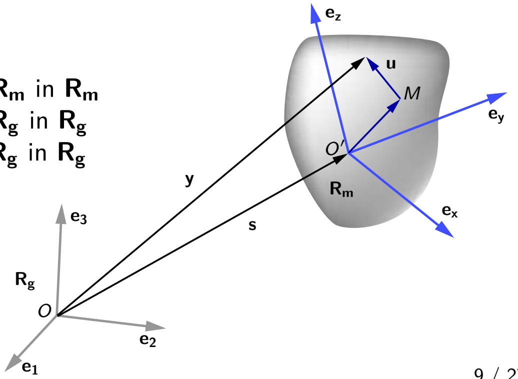
# Inertial and moving frames

## Full 3D rotating motion of the dynamic frame

- Fixed (galilean, inertial) reference frame $\mathbf{R_g}$
- Moving (non-inertial) reference frame $\mathbf{R_m}$
- Rigid body large displacements $\mathbf{R_m}/\mathbf{R_g}$
- Small displacements and strains in $\mathbf{R_m}$

## Coordinates

- $\mathbf{x}$: position vector of $M$ in $\mathbf{R_m}$
- $\mathbf{u}$: displacement vector of $M/\mathbf{R_m}$ in $\mathbf{R_m}$
- $\mathbf{s}$: displacement vector of $O'/\mathbf{R_g}$ in $\mathbf{R_g}$
- $\mathbf{y}$: displacement vector of $M/\mathbf{R_g}$ in $\mathbf{R_g}$



# Displacements and velocities

## Rotation matrices

$$\mathbf{R}_1 = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix};\ \mathbf{R}_2 = \begin{bmatrix} \cos\theta_Y & 0 & -\sin\theta_Y \\ 0 & 1 & 0 \\ \sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix};\ \mathbf{R}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{R} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1$$

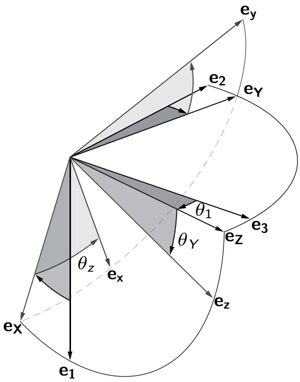
## Coordinates transformation

- absolute displacement $\mathbf{y} = \mathbf{s} + \mathbf{R}(\mathbf{x} + \mathbf{u})$
- angular velocity matrix $\mathbf{\Omega} \mid \dot{\mathbf{R}} = \mathbf{R}\mathbf{\Omega}$

## Velocities

- Time derivatives of positions in $\mathbf{R_g}$
- $\dot{\mathbf{y}} = \dot{\mathbf{s}} + \mathbf{R}\dot{\mathbf{u}} + \dot{\mathbf{R}}(\mathbf{x} + \mathbf{u}) = \dot{\mathbf{s}} + \mathbf{R}\dot{\mathbf{u}} + \mathbf{R}\mathbf{\Omega}(\mathbf{x} + \mathbf{u})$

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -\Omega_3(t) & \Omega_2(t) \\ \Omega_3(t) & 0 & -\Omega_1(t) \\ -\Omega_2(t) & \Omega_1(t) & 0 \end{bmatrix}$$



# Strains and stresses

## Quantities of interest

- Stresses $\boldsymbol{\sigma}$
- Strains $\boldsymbol{\varepsilon}$: Green measure

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right) = \varepsilon_{ij}^{\text{ln}} + \varepsilon_{ij}^{\text{nl}}$$

- Pre-stressed configuration and centrifugal stiffening

## Constitutive law

- Generalized Hooke law $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$
- $\mathbf{C} \rightarrow$ linear elasticity tensor

# Energies and virtual works

## Kinetic energy

$$E_K = \frac{1}{2}\int_{\mathbb{V}} \rho \dot{\mathbf{y}}^T \dot{\mathbf{y}} \mathrm{d}V = \frac{1}{2}\int_{\mathbb{V}} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} \mathrm{d}V + \int_{\mathbb{V}} \rho \dot{\mathbf{u}}^T \mathbf{\Omega} \mathbf{u} \mathrm{d}V + \frac{1}{2}\int_{\mathbb{V}} \rho \mathbf{u}^T \mathbf{\Omega}^2 \mathbf{u} \mathrm{d}V$$

$$- \int_{\mathbb{V}} \rho (\mathbf{u}^T \mathbf{\Omega} - \dot{\mathbf{u}}^T)(\mathbf{R}^T \dot{\mathbf{s}} + \mathbf{\Omega}\mathbf{x}) \mathrm{d}V + \frac{1}{2}\int_{\mathbb{V}} \rho \left( \dot{\mathbf{s}}^T \dot{\mathbf{s}} + 2\dot{\mathbf{s}}^T \mathbf{R}\mathbf{\Omega}\mathbf{x} - \mathbf{x}\mathbf{\Omega}^2 \mathbf{x} \right) \mathrm{d}V$$

## Strain energy

$$U = \frac{1}{2}\int_{\mathbb{V}} \varepsilon^T \mathbf{C} \varepsilon \mathrm{d}V$$

## External virtual works

$$W_{\text{ext}} = \int_{\mathbb{V}} \mathbf{u}^T \mathbf{R} \mathbf{f} \mathrm{d}V + \int_{\mathbb{S}} \mathbf{u}^T \mathbf{R} \mathbf{t} \mathrm{d}S \quad \leftarrow (\mathbf{f}, \mathbf{t}) \text{ expressed in } \mathbf{Rm}$$

# Displacement discretization

## Discretized displacement field

- Rayleigh-Ritz formulation
- Finite Element formulation

$$\Rightarrow \quad \mathbf{u}(\mathbf{x}, t) = \sum_{i=1}^{n} \Phi_i(\mathbf{x}) u_i(t)$$

# Displacement discretization

## Generic equations of motion

$$\mathbf{M}\ddot{\mathbf{u}} + (\mathbf{D} + \mathbf{G}(\boldsymbol{\Omega}))\,\dot{\mathbf{u}} + \Big(\mathbf{K}(\mathbf{u}) + \mathbf{P}(\dot{\boldsymbol{\Omega}}) + \mathbf{N}(\boldsymbol{\Omega})\Big)\,\mathbf{u} = \mathbf{R}(\boldsymbol{\Omega}) + \mathbf{F}$$

## Structural matrices

- mass matrix $\mathbf{M} = \int_{\mathbb{V}} \rho \boldsymbol{\Phi}^T \boldsymbol{\Phi} \mathrm{d}V$
- gyroscopic matrix $\mathbf{G} = \int_{\mathbb{V}} 2\rho \boldsymbol{\Phi}^T \boldsymbol{\Omega} \boldsymbol{\Phi} \mathrm{d}V$
- centrifugal stiffening matrix $\mathbf{N} = \int_{\mathbb{V}} \rho \boldsymbol{\Phi}^T \boldsymbol{\Omega}^2 \boldsymbol{\Phi} \mathrm{d}V$
- angular acceleration stiffening matrix $\mathbf{P} = \int_{\mathbb{V}} \rho \boldsymbol{\Phi}^T \dot{\boldsymbol{\Omega}} \boldsymbol{\Phi} \mathrm{d}V$
- stiffness matrix $\mathbf{K} = \int_{\mathbb{V}} \boldsymbol{\sigma}^T \varepsilon \mathrm{d}V = \int_{\mathbb{V}} \boldsymbol{\varepsilon}^T(\mathbf{u}) \mathbf{C} \boldsymbol{\varepsilon}(\mathbf{u}) \mathrm{d}V$ $\leftarrow$ nonlinear in $\mathbf{u}$
- damping matrix $\mathbf{D}$ (several different definitions)

## External force vectors

- inertial forces $\mathbf{R} = -\int_{\mathbb{V}} \rho \boldsymbol{\Phi}^T (\mathbf{R}^T \ddot{\mathbf{s}} + \dot{\boldsymbol{\Omega}} \mathbf{x} + \boldsymbol{\Omega}^2 \mathbf{x}) \mathrm{d}V$
- external forces $\mathbf{F} = \int_{\mathbb{V}} \boldsymbol{\Phi}^T \mathbf{f} \mathrm{d}V + \int_{\mathbb{S}_F} \boldsymbol{\Phi}^T \mathbf{t} \mathrm{d}S$

# Displacement discretization

## Generic equations of motion

$$\mathbf{M}\ddot{\mathbf{u}} + (\mathbf{D} + \mathbf{G}(\boldsymbol{\Omega}))\,\dot{\mathbf{u}} + \Big(\mathbf{K}(\mathbf{u}) + \mathbf{P}(\dot{\boldsymbol{\Omega}}) + \mathbf{N}(\boldsymbol{\Omega})\Big)\,\mathbf{u} = \mathbf{R}(\boldsymbol{\Omega}) + \mathbf{F}$$

## Choice space functions $\boldsymbol{\Phi}$ and time contributions $\mathbf{u}$

- order-reduced models or modal reductions
- 1D, 2D or 3D elasticity
- beam and shell theories (Euler-Bernoulli, Timoshenko, Reissner-Mindlin...)
- complex geometries: bladed-disks

## Other simplifying assumptions

- constant velocity along a single axis of rotation
- no gyroscopic terms
- no centrifugal stiffening
- localized vs. distributed imbalance
- isotropy or anisotropy of the shaft cross-section

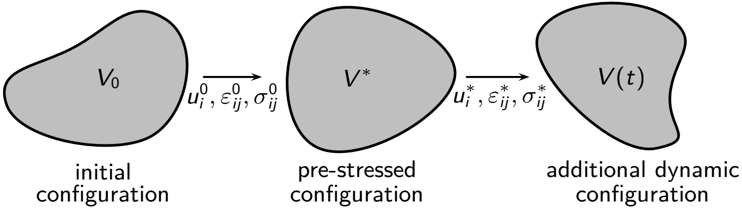
# Outline

# Static equilibrium

## Solution and pre-stressed configuration

Find displacement $\mathbf{u}^0$, solution of:

$$\left(\mathbf{K}(\mathbf{u}^0) + \mathbf{N}(\boldsymbol{\Omega})\right)\mathbf{u}^0 = \mathbf{R}(\boldsymbol{\Omega}) + \mathbf{F}$$




## Applications

- slender structures
- plates, shells
- shaft with axial loading
- radial traction and blade untwist

# Modal analysis

## Eigensolutions

- Defined with respect to the pre-stressed configuration $V^*$
- State-space formulation

$$\begin{bmatrix} \mathbf{M} & \mathbf{D} + \mathbf{G}(\Omega) \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{u}} \\ \dot{\mathbf{u}} \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{K}(\mathbf{u}^0) + \mathbf{N}(\Omega) \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{u}} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{1}$$

- Consider:

$$\mathbf{y} = \begin{pmatrix} \dot{\mathbf{u}} \\ \mathbf{u} \end{pmatrix} = \mathbf{A}\mathrm{e}^{\lambda t} \quad \text{and find } (\mathbf{y}, \lambda) \text{ solution of (1)}$$

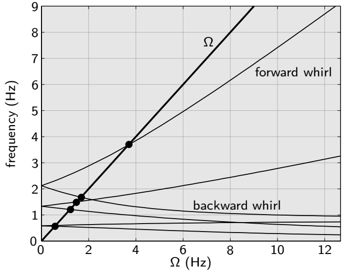
## Comments

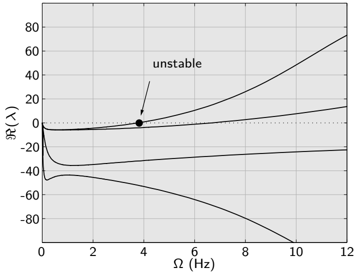
- $\Omega$-dependent complex (conjugate) eigenmodes and eigenvalues
- Right and left eigenmodes
- Whirl motions
- **Campbell diagrams and stability conditions**

# Modal analysis

## Campbell diagrams

- Coincidence of vibration sources and $\omega$-dependent natural resonances
- Quick visualisation of vibratory and design problems




## Stability

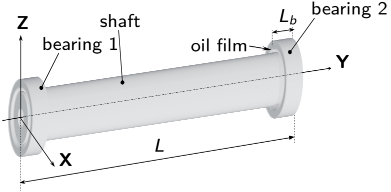
$$\exists \lambda_i \mid \Re(\lambda_i) > 0 \quad \Rightarrow \quad \text{unstable}$$

# Outline

# Case 1: flexible shaft bearing system

## Rotating shaft bearing system

- Constant velocity
- Small deflections
- Elementary slice ⇔ rigid disk




## Quantities of interest

- Kinetic Energy

$$E_K = \frac{\rho S}{2}\int_0^L (\dot{u}^2+\dot{w}^2)\mathrm{d}y + \frac{\rho I}{2}\int_0^L \left[\left(\frac{\partial \dot{u}}{\partial y}\right)^2 + \left(\frac{\partial \dot{w}}{\partial y}\right)^2\right]\mathrm{d}y + \rho I L \Omega^2 - 2\rho I \Omega \int_0^L \frac{\partial \dot{u}}{\partial y}\frac{\partial w}{\partial y}\mathrm{d}y$$

- Strain Energy $U = \frac{EI}{2}\int_0^L \left[\left(\frac{\partial^2 u}{\partial y^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]\mathrm{d}y$

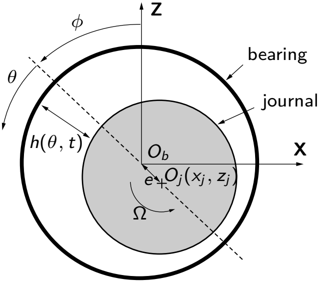
- Virtual Work of external forces $\delta W = F\delta u + G\delta w$

# Case 1: flexible shaft bearing system

## Reynolds' equation

- $h$ journal/bearing clearance
- $\mu$ fluid viscosity
- $\Omega$ angular velocity of the journal
- $R$ outer radius of the beam
- $p$ film pressure

$$\frac{\partial}{\partial y}\left(\frac{h^3}{6\mu}\frac{\partial p}{\partial y}\right)+\frac{1}{R^2}\frac{\partial}{\partial \theta}\left(\frac{h^3}{6\mu}\frac{\partial p}{\partial \theta}\right)=\Omega\frac{\partial h}{\partial \theta}+2\frac{\partial h}{\partial t}$$




## Bearing forces

- $H_j = 1 - Z_j\cos(\theta+\phi) + X_j\sin(\theta+\phi)$
- $G = Z_j\sin(\theta+\phi) + X_j\cos(\theta+\phi) - 2\left(\dot{Z}_j\cos(\theta+\phi) - \dot{X}_j\sin(\theta+\phi)\right)$

$$F_x = -\frac{\mu R {L_b}^3\Omega}{2c^2}\int_0^{\pi}\frac{G\sin(\theta+\phi)}{{H_j}^3}\mathrm{d}\theta; \qquad F_z = \frac{\mu R {L_b}^3\Omega}{2c^2}\int_0^{\pi}\frac{G\cos(\theta+\phi)}{{H_j}^3}\mathrm{d}\theta$$

# Case 1: flexible shaft bearing system

## Linearization

- Equilibrium position $X_e, Z_e$
- First order Taylor series of the nonlinear forces

$$K_{XX} = \left.\frac{\partial F_X}{\partial X}\right|_e ; \quad K_{XZ} = \left.\frac{\partial F_X}{\partial Z}\right|_e ; \quad K_{ZX} = \left.\frac{\partial F_Z}{\partial X}\right|_e ; \quad K_{ZZ} = \left.\frac{\partial F_Z}{\partial Z}\right|_e$$

$$C_{XX} = \left.\frac{\partial F_X}{\partial \dot{X}}\right|_e ; \quad C_{XZ} = \left.\frac{\partial F_X}{\partial \dot{Z}}\right|_e ; \quad C_{ZX} = \left.\frac{\partial F_Z}{\partial \dot{X}}\right|_e ; \quad C_{ZZ} = \left.\frac{\partial F_Z}{\partial \dot{Z}}\right|_e$$

## Field of displacement

$$U(x,t) = \sum_{i=1}^{N} \phi_i(x) u_i(t); \quad W(x,t) = \sum_{i=1}^{N} \phi_i(x) w_i(t) \quad \Rightarrow \quad \mathbf{x} = (\mathbf{u}, \mathbf{w})^T$$

## Linear equations of motion

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{G} + \mathbf{C}_b)\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{K}_b)\mathbf{x} = \mathbf{0}$$

# Case 1: flexible shaft bearing system

## Linear equations of motion

$$\mathbf{M}\ddot{\mathbf{x}} + (\Omega\mathbf{G} + \mathbf{C}_b(\Omega))\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{K}_b(\Omega))\mathbf{x} = \mathbf{F}_{\mathrm{NL}}$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{A}+\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}+\mathbf{B} \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} \mathbf{0} & -2\mathbf{B} \\ 2\mathbf{B} & \mathbf{0} \end{bmatrix}; \quad \mathbf{K} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

$$\mathbf{A} = \rho S \int_0^L \mathbf{\Phi}^T \mathbf{\Phi} \mathrm{d}y; \quad \mathbf{B} = \rho I \int_0^L \mathbf{\Phi}_{,y}^T \mathbf{\Phi}_{,y} \, \mathrm{d}y; \quad \mathbf{C} = EI \int_0^L \mathbf{\Phi}_{,yy}^T \mathbf{\Phi}_{,yy} \, \mathrm{d}y$$

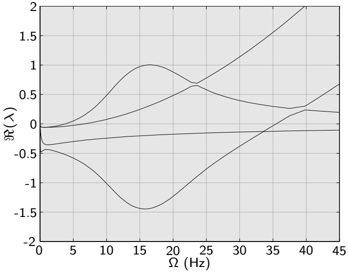
## Comments

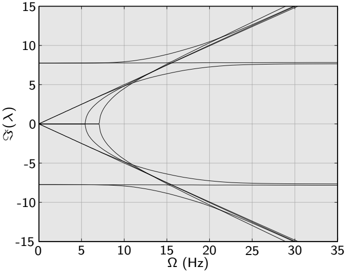
- $\mathbf{K}_b$ circulatory matrix $\rightarrow$ destabilizes the system
- $\mathbf{C}_b$ symmetric damping matrix
- both matrices act on the shaft at the bearing locations (external virtual work)

# Case 1: flexible shaft bearing system

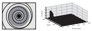
## Modal analysis

- Equilibrium position for each $\Omega$
- $\mathbf{C}_b$, $\mathbf{K}_b$ for each $\Omega$
- Modal analysis for each $\Omega$




## Stability: linear vs. nonlinear models



# Case 1: flexible shaft bearing system

## Modal analysis

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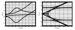

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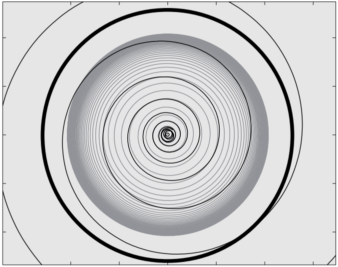
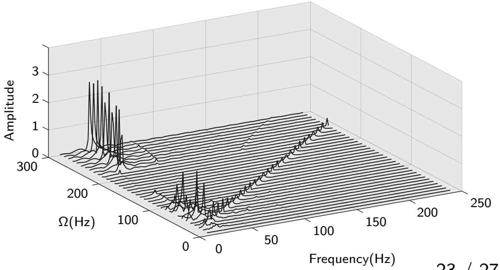
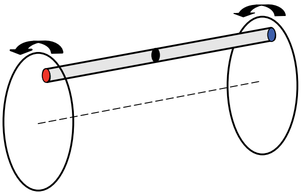


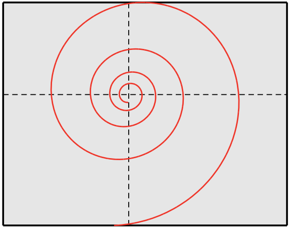


# Case 1: flexible shaft bearing system

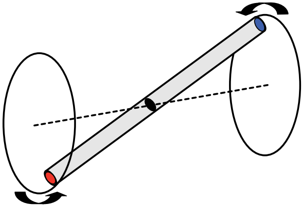
## A few modes

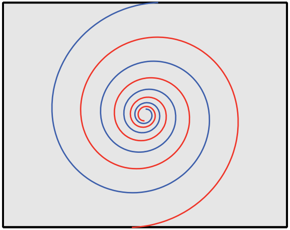
- cylindrical modes



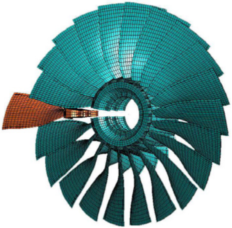


- conical modes





# Case 2: bladed disks



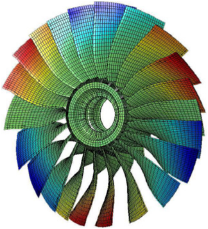
## Definition

- Closed replication of a sector
- Cyclic-symmetry

## Mathematical properties

- Invariant with respect to the axis of rotation
- Block-circulant mass and stiffness matrices
- Pairs of orthogonal modes with identical frequency

## Numerical issues

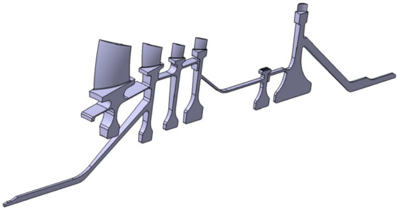


- No damping and gyroscopic matrices
- Determination of **M** and **K**
- Notion of mistuning
- Constant $\Omega$

# Case 2: bladed disks

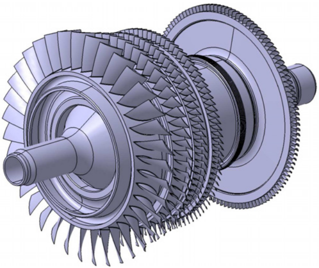
## Multi-stage configuration

- Closed replication of a "slice"
- Different number of blades
- No cyclic-symmetry



## Numerical issues

- No damping and gyroscopic matrices
- Determination of **M** and **K**
- Notion of mistuning
- Constant $\Omega$



## Next-generation calculations

- Nonlinear terms
- Supporting+transmission+operating components
- Damping, gyroscopic terms, stiffening
- Time-dependent $\Omega$

# References

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